

Improving entanglement of even entangled coherent states by a coherent superposition of photon subtraction and addition

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A new entangled quantum state is introduced by applying local coherent superposition ($ra^\dagger + ta$) of photon subtraction and addition to each mode of even entangled coherent state (EECS) and the properties of entanglement are investigated. It is found that the Shchukin-Vogel inseparability, the degree of entanglement and the average fidelity of quantum teleportation of the EECS can be improved due to the coherent superposition operation. The effects of improvement by coherent superposition operation are better than those by single (a^\dagger) and two-photon ($a^\dagger b^\dagger$) addition operations under a small region of amplitude.

Keywords: quantum entanglement, coherent photon addition and subtraction, IWOP method

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I. INTRODUCTION

Quantum entanglement, as an important resource, plays a critical role in quantum computation and quantum information processing [1]. The two-mode squeezed vacuum state exhibits quantum entanglement between the idle mode and the signal mode, and is often used to as an entangled resource. However, the source of practically available is characteristics of an finite degree of entanglement. In order to realize the long-distance communication schemes of quantum information, entanglement resources with higher degree of entanglement are usually required. Thus it is not trivial to enhance the entanglement of given entangled states.

For this purpose, many theoretical and experimental protocols are proposed to generate such entangled states, which are plausible ways to obtain an entangled state with higher entanglement degree by non-Gaussian operations, such as adding or subtracting photons to/from quantum states [2]. For instance, the entanglement between Gaussian entangled states can be increased by subtraction of single photon from Gaussian quadrature-entangled light [3, 4]. The fidelity of quantum teleportation may be enhanced by subtracting one-photon from each mode rather than only one mode [5–7]. In addition, the combination of photon-subtraction and photon-addition operation has been used to generate nonclassical states. For example, nonclassical states of optical field generated by photon subtraction-then-addition or addition-then-subtraction operation [8]. Then multi-photons subtraction-then-addition or addition-then-subtraction is further operated on coherent states. The nonclassicality of these states are discussed according to photon-number distribution, Mandel Q parameter and Wigner function [9]. On the other hand, it is interesting to note that coherent superposition of non-Gaussian operations is also proposed to manipulate and enhance the entanglement of quantum states [10, 11]. In Ref.[10], coherent superposition of photon subtraction and addition, i.e., the $ta + ra^\dagger$ operator is used to transform a classical state to a nonclas-

sical one. Then in Ref. [11] the coherent superposition is further performed on two-mode squeezed vacuum for enhancing quantum entanglement or non-Gaussian entanglement distillation. Recently, the entanglement of a non-Gaussian entangled state is investigated by local photon subtraction operation [12]. Thus these coherent superposition operations can be seen as a useful tool in quantum information processing.

In this paper, we will introduce a kind of entangled state, coherent superposition (CS) even entangled coherent state (EECS) $|\Psi_+(\alpha, 0)\rangle$ (see below Eq.(1)), generated by applying coherent superposition operation $(t_A a + r_A a^\dagger)^m (t_B b + r_B b^\dagger)^n$ to each local mode of two-mode EECSs and investigate the properties of entanglement. The photon-subtraction ($a^m b^n$, $r_A = r_B = 0$) EECS and the photon-addition ($a^\dagger m b^\dagger n$, $r_A = r_B = 1$) EECS can be considered as two special cases of the CS-EECSs. Practically speaking, we investigate the degree of entanglement by using the concurrence, the inseparability by using Shchukin-Vogel (SV) criteria, and the average fidelity of quantum teleportation to explore how these properties can be enhanced by the CS operation. It is shown that all these entanglement characteristics can be improved by operating the CS operation on the EECSs under a certain amplitude region. So far we know, there is no report about this issue in the literature to date.

This paper is arranged as follows. In section 2, we shall introduce the CS-EECSs, and derive the normalization factor by using the generating function of single-variable Hermite polynomials, which is necessary for further discussing the entanglement properties of quantum states. It is shown that the factor is just related to Hermite polynomials. The improvement of entanglement measured by Shchukin-Vogel criteria and the concurrence in sections 3 and 4, respectively. Section 5 is devoted to studying the average fidelity of quantum teleportation with continuous variable by using the CS-EECSs as the entangled channel. It is found that the effective teleportation (fidelity $> 1/2$) can be enhanced by CS operation on the EECSs in a small amplitude region. The main results are involved in the last section.

II. TWO-MODE COHERENT SUPERPOSITION EECSS

In this section, we introduce the two-mode CS-EECSs. First, let us begin with the EECSSs [13], which is defined by

$$|\Psi_+(\alpha, 0)\rangle = N_{+, \alpha} (|\alpha, \alpha\rangle + |-\alpha, -\alpha\rangle), \quad (1)$$

where $|\alpha, \alpha\rangle \equiv |\alpha\rangle_a \otimes |\alpha\rangle_b$ with $|\alpha\rangle_a$ and $|\alpha\rangle_b$ being the usual coherent states in a and b modes, respectively, and $(N_+(\alpha, 0))^{-2} = 2[1 + e^{-4|\alpha|^2}]$ is the normalization constants. By performing repeatedly the two-mode CS operations $\mathfrak{A}_m \equiv (r_A a^\dagger + t_A a)^m$ and $\mathfrak{B}_n \equiv (r_B b^\dagger + t_B b)^n$ on the ECSs, one can obtain the output state as

$$|\Psi_+(\alpha, m, n)\rangle = N_{+, m, n} \mathfrak{A}_m \mathfrak{B}_n (|\alpha, \alpha\rangle + |-\alpha, -\alpha\rangle), \quad (2)$$

where $N_{+, m, n}$ represents the normalization factor and $|t_j|^2 + |r_j|^2 = 1$ ($j = A, B$). In particular, when $t_A = t_B = 0$ and $r_A = r_B = 1$, Eq.(2) just reduces to the two-mode excited EECSSs [14, 15], $|\Psi_+(\alpha, m, n)\rangle = N_{+, m, n} a^{\dagger m} b^{\dagger n} (|\alpha, \alpha\rangle + |-\alpha, -\alpha\rangle)$. The properties of entanglement of single-mode excited EECSSs (say $n = 0$) is discussed in details [16].

Next, we derive the normalization factor $N_{+, m, n}$, which is important for further discussing the properties of statistics and entanglement for quantum states. For this purpose, using the formula $e^{A+B} = e^A e^B e^{-[A, B]/2} = e^B e^A e^{[A, B]/2}$, which is valid for $[A, [A, B]] = [B, [A, B]] = 0$, one can find, for instance,

$$(r_A a^\dagger + t_A a)^m |\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \frac{\partial^m}{\partial s^m} e^{\frac{1}{2}s^2 t_A r_A + \alpha s t_A} e^{(s r_A + \alpha) a^\dagger} |0\rangle \Big|_{s=0}, \quad (3)$$

which leads to (for simplicity, t_i and r_i are considered as real)

$$\begin{aligned} A_1 &\equiv \langle \alpha | (r_A a + t_A a^\dagger)^m (r_A a^\dagger + t_A a)^m | \alpha \rangle \\ &= e^{-|\alpha|^2} \frac{\partial^{2m}}{\partial s^m \partial \tau^m} e^{\frac{1}{2}(s^2 + \tau^2) t_A r_A + \alpha s t_A + \alpha^* \tau t_A} \\ &\quad \times \langle 0 | e^{(\tau r_A + \alpha^*) a} e^{(s r_A + \alpha) a^\dagger} | 0 \rangle \Big|_{s, \tau=0} \\ &= \frac{\partial^{2m}}{\partial s^m \partial \tau^m} \exp \left\{ \frac{t_A r_A}{2} (s^2 + \tau^2) \right\} \\ &\quad \times \exp (\tau s r_A^2 + s (t_A \alpha + r_A \alpha^*) + \tau (t_A \alpha^* + r_A \alpha)) \Big|_{s, \tau=0}. \end{aligned} \quad (4)$$

Using the following formula [17]

$$\begin{aligned} &\frac{\partial^{2m}}{\partial t^m \partial s^m} \exp [R_1 s + R_1^* t + R_3 t s + R_4 (t^2 + s^2)] \Big|_{t=s=0} \\ &= \sum_{l=0}^m \frac{R_3^l R_4^{m-l} (m!)^2}{l! [(m-l)!]^2} \left| H_{m-l} [R_1 / (2i\sqrt{R_4})] \right|^2, \end{aligned} \quad (5)$$

where $H_m(x)$ is the single-variable Hermite polynomials, one can put Eq.(4) into the following form

$$A_1 = \sum_{l=0}^m \frac{(m!)^2 t_A^{m-l} r_A^{m+l}}{2^{m-l} l! [(m-l)!]^2} |H_{m-l} [\bar{\alpha}_A]|^2. \quad (6)$$

Here we have set $\bar{\alpha}_A = (t_A \alpha + r_A \alpha^*) / (i\sqrt{2t_A r_A})$. In a similar way, one can derive

$$\begin{aligned} A_2 &\equiv \langle -\alpha | (r_A a + t_A a^\dagger)^m (r_A a^\dagger + t_A a)^m | \alpha \rangle \\ &= e^{-2|\alpha|^2} \sum_{l=0}^m \frac{(-1)^{m-l} (m!)^2 t_A^{m-l} r_A^{m+l}}{2^{m-l} l! [(m-l)!]^2} |H_{m-l} [\bar{\beta}_A]|^2, \end{aligned} \quad (7)$$

and

$$\begin{aligned} B_1 &\equiv \langle \alpha | (r_B b + t_B b^\dagger)^n (r_B b^\dagger + t_B b)^n | \alpha \rangle \\ &= \sum_{l=0}^n \frac{(n!)^2 t_B^{n-l} r_B^{n+l}}{2^{n-l} l! [(n-l)!]^2} |H_{n-l} [\bar{\alpha}_B]|^2, \end{aligned} \quad (8)$$

as well as

$$\begin{aligned} B_2 &\equiv \langle -\alpha | (r_B b + t_B b^\dagger)^n (r_B b^\dagger + t_B b)^n | \alpha \rangle \\ &= e^{-2|\alpha|^2} \sum_{l=0}^n \frac{(-1)^{n-l} (n!)^2 t_B^{n-l} r_B^{n+l}}{2^{n-l} l! [(n-l)!]^2} |H_{n-l} [\bar{\beta}_B]|^2, \end{aligned} \quad (9)$$

where $\bar{\beta}_A = (r_A \alpha - \alpha^* t_A) / (i\sqrt{2t_A r_A})$, $\bar{\alpha}_B = (t_B \alpha + r_B \alpha^*) / (i\sqrt{2t_B r_B})$ and $\bar{\beta}_B = (r_B \alpha - \alpha^* t_B) / (i\sqrt{2t_B r_B})$.

Thus the normalization factor $N_{+, m, n}$ can be given by

$$(N_{+, m, n})^{-2} = 2 (A_1 B_1 + A_2 B_2). \quad (10)$$

It is noted that when $m = n = 0$, then $A_1 = B_1 = 1$, and $A_2 = B_2 = e^{-2|\alpha|^2}$, thus Eq.(2) becomes the usual EECSSs in Eq.(1) with $N_{+, 0, 0}^{-2} = 2[1 + \exp(-4|\alpha|^2)]$. For the case of $t_A = t_B = 0$ and $r_A = r_B = 1$, corresponding to the two-mode excited EECSSs, from Eqs.(4)-(9), one can see that

$$A_1 = m! L_m(-|\alpha|^2), B_1 = n! L_n(-|\alpha|^2), \quad (11)$$

and

$$\begin{aligned} A_2 &= m! \exp(-2|\alpha|^2) L_m(|\alpha|^2), \\ B_2 &= n! \exp(-2|\alpha|^2) L_n(|\alpha|^2), \end{aligned} \quad (12)$$

thus the normalization factor Eq.(10) reduces to

$$\begin{aligned} (N_{+, m, n})^{-2} &= 2m!n! [L_m(-|\alpha|^2) L_n(-|\alpha|^2) \\ &\quad + e^{-4|\alpha|^2} L_m(|\alpha|^2) L_n(|\alpha|^2)], \end{aligned} \quad (13)$$

where $L_m(x)$ is the m -order Laguerre polynomial. Eq.(13) is just the normalization factor of the two-mode excited EECSSs.

III. SHCHUKIN-VOGEL INSEPARABILITY

For a bipartite continuous-variable state, here we will take the Shchukin-Vogel (SV) criteria [18] to describe the

inseparability of the CS-EECSs. According to the SV criteria, the sufficient condition of inseparability (entanglement) is given by [19]

$$S_+ \equiv \left\langle a^\dagger a - \frac{1}{2} \right\rangle \left\langle b^\dagger b - \frac{1}{2} \right\rangle - \langle a^\dagger b^\dagger \rangle \langle ab \rangle < 0. \quad (14)$$

Thus we only need to calculate three average values $\langle a^\dagger a \rangle$, $\langle b^\dagger b \rangle$, $\langle ab \rangle$ to obtain S_+ .

Here, we consider the property of entanglement of the CS-EECSs with $m = n = 1$, $|\Psi_+(\alpha, 1, 1)\rangle = N_{+,1,1} \mathfrak{A}_1 \mathfrak{B}_1 (|\alpha, \alpha\rangle + |-\alpha, -\alpha\rangle)$, where $N_{+,1,1}^{-2} = 2(A_1 B_1 + A_2 B_2)$ and

$$\begin{aligned} A_1 &= |t_A \alpha + r_A \alpha^*|^2 + r_A^2, \\ B_1 &= |t_B \alpha + r_B \alpha^*|^2 + r_B^2, \\ A_2 &= (r_A^2 - |r_A \alpha - \alpha^* t_A|^2) e^{-2|\alpha|^2}, \\ B_2 &= (r_B^2 - |r_B \alpha - \alpha^* t_B|^2) e^{-2|\alpha|^2}. \end{aligned} \quad (15)$$

Under the state $|\Psi_+(\alpha, 1, 1)\rangle$, we can derive three average values as

$$\begin{aligned} \langle a^\dagger a \rangle_{1,1} &= \frac{O_{A1} B_1 + O_{A2} B_2}{A_1 B_1 + A_2 B_2}, \\ \langle b^\dagger b \rangle_{1,1} &= \frac{O_{B1} A_1 + O_{B2} A_2}{A_1 B_1 + A_2 B_2}, \\ \langle ab \rangle_{1,1} &= \frac{R_{A1} R_{B1} + R_{A2} R_{B2}}{A_1 B_1 + A_2 B_2}, \end{aligned} \quad (16)$$

where $A_{1,2}$ and $B_{1,2}$ are defined in Eqs.(15) and we have set

$$\begin{aligned} O_{j1} &= |\alpha|^4 + r_j^2(1 + 3|\alpha|^2) \\ &\quad + t_j r_j (\alpha^{*2} + \alpha^2)(1 + |\alpha|^2), \\ O_{j2} &= e^{-2|\alpha|^2} \left[|\alpha|^4 + r_j^2(1 - 3|\alpha|^2) \right. \\ &\quad \left. + t_j r_j (\alpha^{*2} + \alpha^2)(1 - |\alpha|^2) \right], \end{aligned} \quad (17)$$

and

$$\begin{aligned} R_{j1} &= |\alpha|^2 \alpha + t_j r_j (\alpha^3 + |\alpha|^2 \alpha^* + \alpha^*) + 2r_j^2 \alpha, \\ R_{j2} &= \left[-|\alpha|^2 \alpha + t_j r_j (\alpha^3 + |\alpha|^2 \alpha^* - \alpha^*) + 2r_j^2 \alpha \right] e^{-2|\alpha|^2}, \end{aligned} \quad (18)$$

$$(j = A, B).$$

Using Eqs.(15)-(18) we can get the expression of S_+ . Here we appeal to the numerical calculation for further discussion. In Fig. 2, we plot the sufficient condition of inseparability as the function of α (as a real number) for the EECs and the CS-EECs with several different values of t . It is shown that the condition $S_+ < 0$ can be satisfied only when α exceeds a certain threshold value

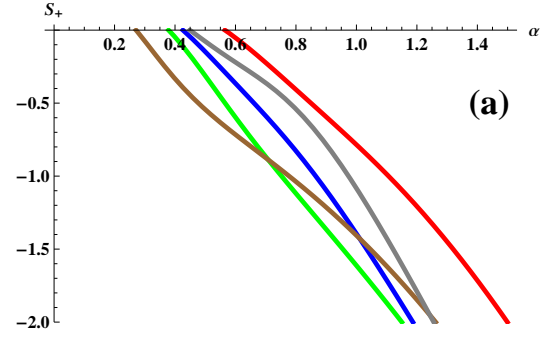


FIG. 1: (Color online) The sufficient condition of inseparability as the function of α for the EECs (red line) and CS-EECs with several different values of t : $t=0$ (Gray line), $t=0.2$ (Blue line), $t=0.6$ (Green line) and $t=0.9$ (Brown).

($\alpha > 0.567$ about). In fact, under the EECs we have $\langle a^\dagger a \rangle = \langle b^\dagger b \rangle = |\alpha|^2 \tanh(2|\alpha|^2)$, and $\langle ab \rangle = \alpha^2$, thus

$$S_+ = \left(|\alpha|^2 \tanh(2|\alpha|^2) - \frac{1}{2} \right)^2 - |\alpha|^4 < 0, \quad (19)$$

which lead to the following condition:

$$2|\alpha|^2 \left(\tanh(2|\alpha|^2) + 1 \right) > 1. \quad (20)$$

For the coherent superposition operation ($0 < t < 1$), to satisfy the SV criteria, the value of α must be larger than a certain threshold value which decreases with the increasing t (For simplicity we take $t_{A,B} = t$ and $r_{A,B} = r$, α as real). From Fig.1, we can see that (i) these threshold values in CS-EECs are smaller than that in EECs; (ii) the inseparability in the CS-EECs is stronger than in the EECs and the photon-added EECs ($a^\dagger b^\dagger |\Psi_+(\alpha, 0)\rangle$), at $t = 0$; This indicates that the coherent operation on both local modes improves the inseparability better than photon-addition operation $a^\dagger b^\dagger$ in the small amplitude region. (iii) the photon-subtraction operation ab (at $t = 1$) can not be used to enhance the inseparability of the EECs $|\Psi_+(\alpha, 0)\rangle$ due to the fact that $|\Psi_+(\alpha, 0)\rangle$ is the eigenstate of pair annihilation operator ab . From the point of SV criteria, the inseparability of the EECs can be improved better by operating CS on both local modes than photon-addition operation ($a^\dagger b^\dagger$) under a certain condition.

IV. DEGREE OF ENTANGLEMENT OF THE CS-EECS

Next, we calculate the amount of entanglement of the CS-EECs and investigate the influence of the two-mode coherent superposition operation on the entanglement of the CS-EECs. For continuous-variables-type entangled states like (2), the degree of quantum entanglement of the bipartite entangled states can be measured in terms of the concurrence. Following the approach [20–22], for

two systems involving nonorthogonal states defined as

$$|\Psi\rangle = N[\mu|\eta\rangle_a \otimes |\gamma\rangle_b + \nu|\xi\rangle_a \otimes |\delta\rangle_b], \quad (21)$$

where $|\eta\rangle_a$, $|\xi\rangle_a$ and $|\gamma\rangle_b$, $|\delta\rangle_b$ are normalized states of system a and b , respectively, the concurrence can be calculated as [23, 24]

$$C = \frac{2|\mu||\nu| \left[(1-|p_1|^2)(1-|p_2|^2) \right]^{1/2}}{|\mu|^2 + |\nu|^2 + 2\text{Re}(\mu^* \nu p_1 p_2^*)}, \quad (22)$$

where the normalization constant is $N^{-2} = |\mu|^2 + |\nu|^2 + 2\text{Re}(\mu^* \nu p_1 p_2^*)$, with complex numbers μ and ν , and $p_1 = {}_a\langle\eta|\xi\rangle_a$, $p_2 = {}_b\langle\delta|\gamma\rangle_b$. When $\mu = \nu$, the concurrence C becomes

$$C_+ = \frac{\sqrt{(1-|p_1|^2)(1-|p_2|^2)}}{1 + \text{Re}(p_1 p_2^*)}, \quad (23)$$

which only dependent on the overlaps $\langle\eta|\xi\rangle$ and $\langle\gamma|\delta\rangle$.

In order to obtain the amount of entanglement of the CS-ECSs, we introduce the following normalized states:

$$\begin{aligned} |\pm\alpha\rangle_m &= A_1^{-1/2} (r_A a^\dagger + t_A a)^m |\pm\alpha\rangle, \\ |\pm\alpha\rangle_n &= B_1^{-1/2} (r_B b^\dagger + t_B b)^n |\pm\alpha\rangle, \end{aligned} \quad (24)$$

then the CS-ECSs $|\Psi_+(\alpha, m, n)\rangle$ in terms of four normalized states can be reformed as

$$|\Psi_+(\alpha, m, n)\rangle = N_{+,m,n} (A_1 B_1)^{1/2} (|\alpha\rangle_m \otimes |\alpha\rangle_n + |-\alpha\rangle_m \otimes |-\alpha\rangle_n). \quad (25)$$

Thus one can find

$$\begin{aligned} p_1 &= {}_m\langle\alpha|-\alpha\rangle_m = \frac{A_2}{A_1}, \\ p_2 &= {}_n\langle-\alpha|\alpha\rangle_n = \frac{B_2}{B_1}. \end{aligned} \quad (26)$$

Substituting Eq.(26) into Eq.(23) one can get

$$C_+ = \frac{\sqrt{(A_1^2 - A_2^2)(B_1^2 - B_2^2)}}{A_1 B_1 + A_2 B_2}, \quad (27)$$

where A_1, B_1 and A_2, B_2 are defined in Eqs.(6)-(9). Eq.(27) is the analytical expression of concurrence for the CS-EECSSs.

In particular, for the two-mode excited EECSSs, i.e., $t_A = t_B = 0$ and $r_A = r_B = 1$, leading to $A_1 = L_m(-|\alpha|^2)$, $B_1 = L_n(-|\alpha|^2)$ and $A_2 = e^{-2|\alpha|^2} L_m(|\alpha|^2)$, $B_2 = e^{-2|\alpha|^2} L_n(|\alpha|^2)$, then Eq.(27) reduces to

$$\begin{aligned} C_+ &= \frac{\sqrt{L_m^2(-|\alpha|^2) - e^{-4|\alpha|^2} L_m^2(|\alpha|^2)}}{L_m(-|\alpha|^2) L_n(-|\alpha|^2) + e^{-4|\alpha|^2} L_m(|\alpha|^2) L_n(|\alpha|^2)} \\ &\times \sqrt{L_n^2(-|\alpha|^2) - e^{-4|\alpha|^2} L_n^2(|\alpha|^2)}, \end{aligned} \quad (28)$$

which exhibits the exchanging symmetry with respect to m and n , namely, $C_+(\alpha, n, m) = C_+(\alpha, m, n)$. In addition, when $m = n$, Eq.(28) becomes

$$C_+ = \frac{L_m^2(-|\alpha|^2) - e^{-4|\alpha|^2} L_m^2(|\alpha|^2)}{L_m^2(-|\alpha|^2) + e^{-4|\alpha|^2} L_m^2(|\alpha|^2)}, \quad (29)$$

which implies that the two-mode symmetrically excited EECSSs is also not a maximally entangled state. As expected, the concurrence of single-mode excited EECSSs is just a special case of Eq.(28) [16]. For the two-mode subtraction ($t_A = t_B = 1$ and $r_A = r_B = 0$), the concurrence C_+ keep unchanged for the total even number photon-subtraction case ($m + n = \text{even number}$); while for the total odd number photon-subtraction case ($m + n = \text{odd number}$), the states shall present a transformation from the partially entangled state to the maximum entangled state.

In order to see clearly the effect of different parameters r_j, t_j and (m, n) on the concurrence of CS-EECSSs, C_+ for the state $|\Psi_+(\alpha, m, n)\rangle$ as a function of different parameters are shown in Figs.2-3. We show the concurrence C_+ with a symmetric case of $r_A = r_B = r$ and $m = n = 1, 2$ in Fig.2, from which we can see that the concurrence C_+ increases with r (α) when α (r) exceeds a threshold; while the case is still true for $m = n = 2$ but without the threshold. Here, for convenience, we plotted only with α being real number.

From Fig.3(a)-(b) with a given value of $r = 1/\sqrt{2}$, one can see that C_+ increases with the increase of α for given parameters m and n . Especially, the concurrence C_+ tends to unit for the larger α . In addition, for a symmetric case of $m = n$, C_+ increases with the increase of m (Fig.3(a)); while for an asymmetric case of $m \neq n$, C_+ increases with the increase of n (Fig.3(b)). Thus the (high-order) coherent superposition operation can be applied to enhance the entanglement of the state $|\Psi_+(\alpha, m, n)\rangle$.

In particular, we make a comparison of the entanglement properties between the CS-EECSSs (single-mode CS-EECSSs $|\Psi_+(\alpha, 1, 0)\rangle$ and $|\Psi_+(\alpha, 1, 1)\rangle$) and the single(two)-mode photon excited EECSSs ($a^\dagger |\Psi_+(\alpha, 0, 0)\rangle$ and $a^\dagger b^\dagger |\Psi_+(\alpha, 0, 0)\rangle$) (see Fig.4). From Fig.4 one can clearly see that: (i) both photon-excitation and CS operations can improve the entanglement of $|\Psi_+(\alpha, 0, 0)\rangle$; (ii) for single-mode operations ($ta + ra^\dagger$ and a^\dagger), the effects of the CS operations are more prominent than those of the mere photon-addition in a small-amplitude regime. The threshold value is about $\alpha \lesssim 0.6$ with a symmetrical case of $r = 1/\sqrt{2}$. In addition, the improvement of entanglement is more obvious by an asymmetrical parameter (say $r = 0.4$) than that by $r = 1/\sqrt{2}$. These results show that applying a CS operator on the EECSSs may be better to enhance the quantum entanglement than using only a creation operator on the EECSSs, especially for an asymmetrical parameter r . For two-mode operations ($(ta + ra^\dagger)(tb + rb^\dagger)$ and $a^\dagger b^\dagger$), the case is true (However, we should point that the most optimal case is not

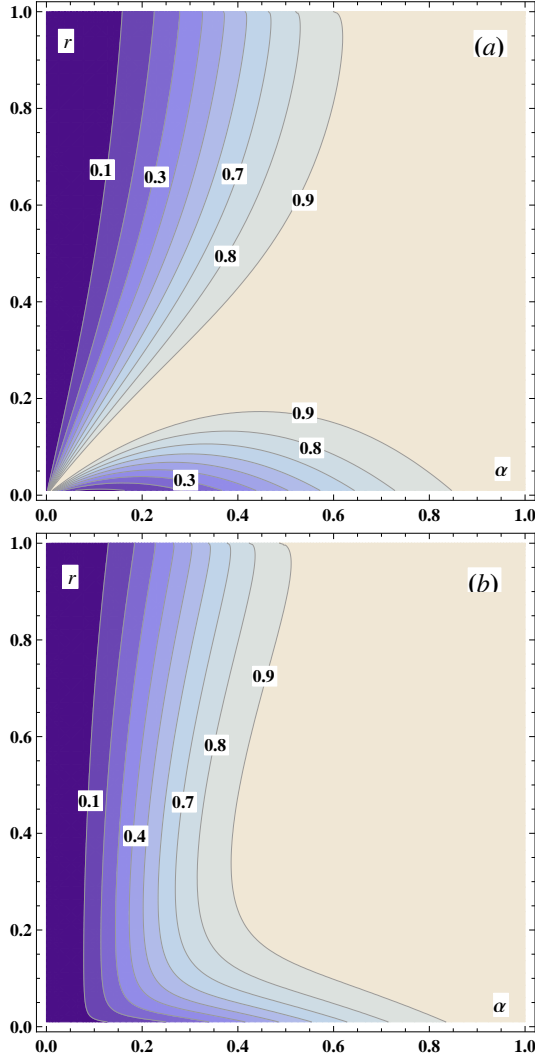


FIG. 2: (Color online) Concurrence of entanglement of $|\Psi_+(\alpha, m, n)\rangle$ as a function of α (considered as a real number) and r for different m and n values. (a) $m = n = 1$; (b) $m = n = 2$.

for $r = 0$ or $t = 1$ because two-mode photon-subtraction EECs $ab|\Psi_+(\alpha, 0, 0)\rangle$ is $|\Psi_+(\alpha, 0, 0)\rangle$. (iii) Moreover, the two-mode operations are more effective on increasing the entanglement of EECs than the single-mode operations. The entanglement can be modulated by different values of r , which can also be clearly seen from Fig.3(a).

V. FIDELITY OF QUANTUM TELEPORTATION

It is well known that photon subtraction from or photon addition to bipartite Gaussian states can be used to improve the entanglement and quantum teleportation [3, 4, 11, 25–32]. In this section, we consider the effect of coherent superposition on quantum teleportation (QT) by using the CS-EECs as entangled resource. In

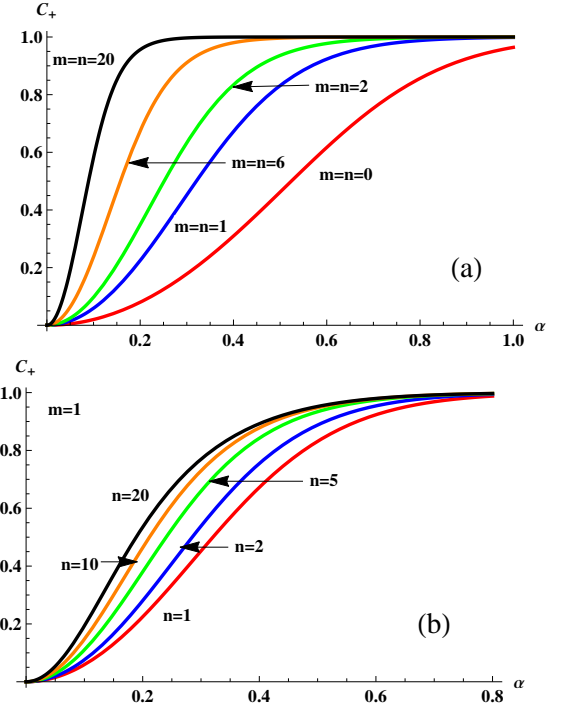


FIG. 3: (Color online) Concurrence of entanglement for the CS-EECs $|\Psi_+(\alpha, m, n)\rangle$ with $r = 1/\sqrt{2}$. (a) $m = n$; (b) $m \neq n$.

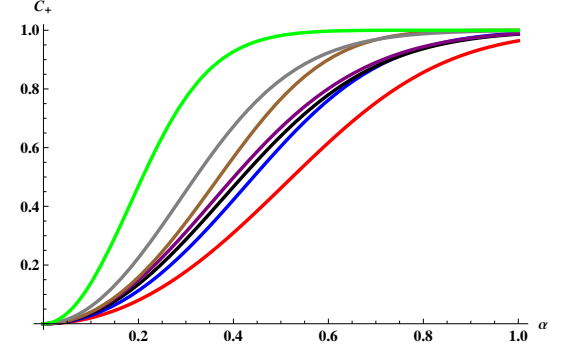


FIG. 4: (Color online) Concurrence of entanglement for the states: ECSs $|\Psi_+(\alpha, 0, 0)\rangle$ (Red line), single-photon excited ECSs $a^\dagger|\Psi_+(\alpha, 0, 0)\rangle$ (Blue line), single-mode CS-ECSs $|\Psi_+(\alpha, 1, 0)\rangle$ (Black and Purple lines with $r = 1/\sqrt{2}, 0.6$ respectively), two-mode excited CESs $a^\dagger b^\dagger|\Psi_+(\alpha, 0, 0)\rangle$ (Origin line), two-mode CS-CESs $|\Psi_+(\alpha, 1, 1)\rangle$ (Gray and Green lines with $r = 1/\sqrt{2}, 0.4$ respectively).

order to describe the quality of a QT scheme, the fidelity between a unknown input state and the teleported output state is usually used as a measure. For a continuous-variable (CV) system, a teleportation scheme has been proposed according to the characteristic functions (CFs) of the quantum states concluding input, source and teleported states [33].

For a two-mode system ρ , the CF is defined as $\chi(\alpha, \beta) = \text{tr}[D_a(\eta)D_b(\gamma)\rho]$, where $D_{a,b}$ are the dis-

placement operators. Noticing the displacement operator $D_a(\eta) = e^{-|\eta|^2/2} e^{\eta a^\dagger} e^{-\eta^* a}$ and using (3), then the CF of the CS-EECs can be calculated as

$$\begin{aligned} \chi_E(\eta, \gamma) = & N_{\pm, m, n}^2 \{ CF_A^{\alpha, \alpha}(\eta) CF_B^{\alpha, \alpha}(\gamma) \\ & + CF_A^{-\alpha, -\alpha}(\eta) CF_B^{-\alpha, -\alpha}(\gamma) \\ & \pm e^{-4|\alpha|^2} CF_A^{\alpha, -\alpha}(\eta) CF_B^{\alpha, -\alpha}(\gamma) \\ & \pm e^{-4|\alpha|^2} CF_A^{-\alpha, \alpha}(\eta) CF_B^{-\alpha, \alpha}(\gamma) \}, \end{aligned} \quad (30)$$

where we have set (noticing $(\beta, \alpha) = (\pm\alpha, \pm\alpha)$)

$$\begin{aligned} CF_j^{\beta, \alpha}(\eta) &= e^{-|\eta|^2/2 + \eta\beta^* - \eta^*\alpha} \frac{\partial^{2m}}{\partial \tau^m \partial s^m} e^{\frac{1}{2}(\tau^2 + s^2)t_j r_j} \\ &\times e^{(\eta r_j + r_j \alpha + \beta^* t_j)\tau + (\alpha t_j - \eta^* r_j + r_j \beta^*)s + \tau s r_j^2} \Big|_{s, \tau=0}, \end{aligned} \quad (31)$$

$(j = A, B).$

For convenience of further calculation, here we keep the differential form of $CF_j^{\beta, \alpha}(\eta)$ in Eq.(31).

Next, we consider the Braunstein and Kimble protocol [34] of QT for single-mode coherent-input states $|\gamma\rangle$. Note that the fidelity is independent of amplitude of the coherent state, thus for simplicity we take $\gamma = 0$, then we have only to calculate the fidelity of the vacuum input state with the CF $\chi_{in}(z) = \exp[-|z|^2/2]$. It is shown that, with the CF $\chi_E(\eta, \gamma)$ for entangled channel, the CF $\chi_{out}(z)$ of the output state can be related to the CFs of input state and entangled source by formula $\chi_{out}(z) = \chi_{in}(z) \chi_E(z^*, z)$, and the fidelity of QT of CVs can be obtained as [33]

$$\mathcal{F} = \int \frac{d^2 z}{\pi} \chi_{in}(z) \chi_{out}(-z). \quad (32)$$

Thus substituting Eqs.(30)-(31) into Eq. (32) yields

$$\begin{aligned} \mathcal{F}_{m, n}^+ = & N_{+, m, n}^2 [\mathcal{F}^{\alpha, \alpha} + \mathcal{F}^{-\alpha, -\alpha} \\ & + e^{-4|\alpha|^2} (\mathcal{F}^{\alpha, -\alpha} + \mathcal{F}^{-\alpha, \alpha})], \end{aligned} \quad (33)$$

where we have derived

$$\begin{aligned} \mathcal{F}^{\beta, \alpha} &= \frac{1}{2} e^{\frac{1}{2}(\beta^* - \alpha)^2} \sum_{l=0}^m \sum_{f=0}^n \sum_{k, j=0}^{\min(n-f, m-l)} \\ &\times \frac{(m!)^2 (n!)^2}{2^{m+n} l! f! k! j!} \frac{(-1)^{k+j} t_A^{m-l} t_B^{n-f}}{(m-l-k)! (m-l-j)!} \\ &\times \frac{r_A^{m+l} r_B^{n+f} (r_A r_B)^{k+j} (\sqrt{t_A r_A t_B r_B})^{-k-j}}{(n-f-k)! (n-f-j)!} \\ &\times H_{m-l-k}(N_2) H_{m-l-j}(N_1) H_{n-f-k}(M_2) H_{n-f-j}(M_1), \end{aligned} \quad (34)$$

and set

$$\begin{aligned} N_1 &= [\beta^* t_A + \frac{1}{2} r_A (\beta^* + \alpha)] / (-i\sqrt{2t_A r_A}), \\ N_2 &= [\alpha t_A + \frac{1}{2} r_A (\beta^* + \alpha)] / (i\sqrt{2t_A r_A}), \\ M_1 &= [\beta^* t_B + \frac{1}{2} r_B (\beta^* + \alpha)] / (-i\sqrt{2t_B r_B}), \\ M_2 &= [\alpha t_B + \frac{1}{2} r_B (\beta^* + \alpha)] / (i\sqrt{2t_B r_B}). \end{aligned} \quad (35)$$

Eq.(33) is just the fidelity of teleporting the coherent state by using the CS-EECs as entangled resource. In particular, when $m = n = 0$, Eq.(34) reduces to $\mathcal{F}^{\beta, \alpha} = \frac{1}{2} e^{\frac{1}{2}(\beta^* - \alpha)^2}$, submitting it into Eq.(33) yields

$$\mathcal{F}_{0,0}^+ = \frac{e^{\frac{1}{2}(\alpha^* - \alpha)^2} + e^{-4|\alpha|^2} e^{\frac{1}{2}(\alpha^* + \alpha)^2}}{2(1 + e^{-4|\alpha|^2})}, \quad (36)$$

which is just the fidelity by using the usual EECs as entangled channel. It depends on the real and imaginary parts of coherent amplitude α of the quantum channel, which is different from the case [35, 36]. If the fidelity exceeds the classical limit 1/2, then the teleportation can be considered as a successful quantum protocol. From Eq.(36) it is found that under a small region of α ($|\alpha| < 2.2$), $\mathcal{F}_{0,0}^+$ may be larger than 1/2 (see Fig.5) (the fidelity is always less than 1/2 for odd entangled coherent state). This implies that although the odd entangled coherent state is the maximum entangled coherent state, the fidelity teleporting coherent state has not been enhanced, and that the non-maximum entangled state can be used as more effective resource for quantum teleportation [28–30].

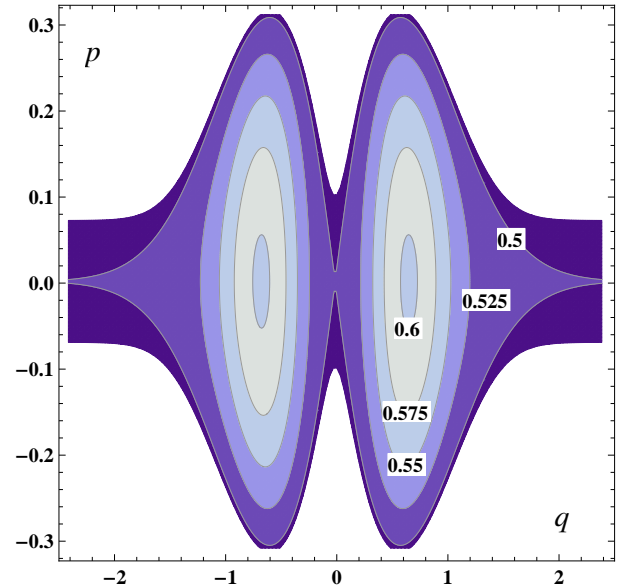


FIG. 5: (Color online) The fidelity $\mathcal{F}_{0,0}^+$ of teleporting coherent state by using EECs as entangled channel, where $\alpha = q + ip$.

In order to know how the CS-EECs can improve the

fidelity of QT, we plot the fidelity of teleporting coherent state by using CS-EECSs as the entangled resource with a symmetric case of $r_A = r_B = r$ ($0 < r < 1$) in Fig.6(a). It is shown that the maximum fidelity for $\mathcal{F}_{1,1}^+$ can be realized by the CS operation in the small region of coherent amplitude α (taking a real number) and the value of θ for optimal fidelity will be modulated by different values of small α . Generally, the fidelity increases with the increase of the value of θ . To obtain an effective QT, i.e., $\mathcal{F}_{1,1}^+ > 1/2$, r must be less than $r = 0.6$. For instance, from Fig.6(a) we can see that the fidelity 0.65 can be obtained well above the classical limit by the CS, say for $r = 0.05$ and $\alpha = 0.1$. This case is true for coherent superposition two-mode squeezed vacuum [11]. In addition, we should mention that the state $|\Psi_+(\alpha, 1, 1)\rangle$ becomes $|\Psi_+(\alpha, 1, 1)\rangle \sim ab|\Psi_+(\alpha, 0, 0)\rangle \sim |\Psi_+(\alpha, 0, 0)\rangle$ at $r = 0$, thus the line of $r = 0$ is just the fidelity by using the EECSs as entangled channel. From this point, we can see that for a given small value of α the fidelity can be optimized over the classical limit and over the fidelity at $r = 0$. For the case of the odd entangled coherent state, there is no any effective fidelity found for any values of r and α .

As a comparison between the CS-EECSs and the EECSs, in Fig.6(b) we display the difference of fidelity ($\Delta\mathcal{F}^+ = \mathcal{F}_{1,1}^+ - \mathcal{F}_{0,0}^+$) as the function of α and r . It is shown that there is an over-zero region for the fidelity difference $\Delta\mathcal{F}^+$ which means that the CS operation can be used to improve the fidelity teleporting coherent state.

In Fig. 7, we plot the fidelity as a function of α for several different values of (m, n) with a symmetric case of $m = n$ and $r = 0.195$. From Fig.7 we can see that the fidelity $\mathcal{F}_{m,m}^+$ decrease with the values of m only when α exceeds a certain threshold value (Fig.7(a)). In particular, $\mathcal{F}_{2,2}^+$ may achieve more optimal value over classical limit than $\mathcal{F}_{1,1}^+$ in a small region of α . On the other hand, for an asymmetric case ($m \neq n$, $r = 0.195$), it is interesting to notice that the fidelity $\mathcal{F}_{m,n}^+$ decreases due to the asymmetric operation (Fig.7(b)). These indicate that multiple CS operations may not only beat the classical bound but also even surpass single CS operations on each mode under a certain condition.

VI. CONCLUSIONS

In this paper, we have introduced the concept of two-mode coherent superposition EECSs by coherent operation ($ra^\dagger + ta$) of photon subtraction and addition, and investigated the properties of entanglement according to the concurrence, the Shchukin-Vogel criteria and the average fidelity of quantum teleportation. It is shown that for the EECS, all of entanglement characteristics can be improved by local coherent operation. We made a comparison of the entanglement properties between the CS-EECSs (single-mode CS-ECSs $|\Psi_+(\alpha, 1, 0)\rangle$ and $|\Psi_+(\alpha, 1, 1)\rangle$) and the single(two)-mode photon excited EECSs ($a^\dagger|\Psi_+(\alpha, 0, 0)\rangle$ and $a^\dagger b^\dagger|\Psi_+(\alpha, 0, 0)\rangle$), which

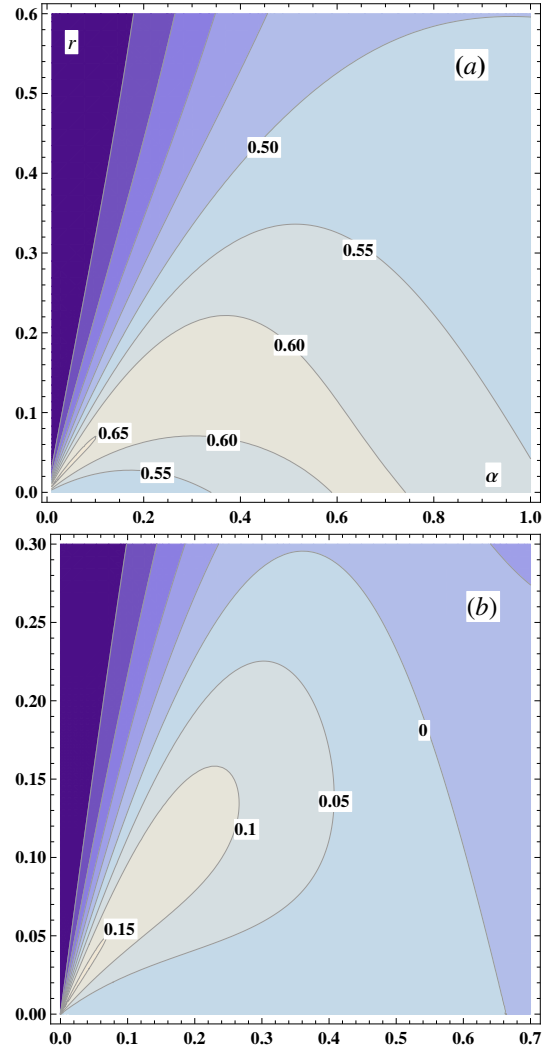


FIG. 6: (Color online) The fidelity of teleporting coherent state by using CS-EECSs as entangled channel, where α is a real number and $r_A = r_B = r$. (a) $\mathcal{F}_{1,1}^+$ as a function α and r ; (b) $\mathcal{F}_{1,1}^+ - \mathcal{F}_{0,0}^+$ as a function α and r ;

shows that the effects of improvement by coherent superposition operation ($ta + ra^\dagger$) ($tb + rb^\dagger$) are more prominent than those by single (a^\dagger) and two-photon ($a^\dagger b^\dagger$) addition under a small region of amplitude, especially for an asymmetrical parameter r . Using the CS-EECSs with a symmetric case ($r_A = r_B = r$) as an entangled channel, the fidelity of teleporting a coherent state is also considered, which presents that more effective quantum teleportation can be realized by $|\Psi_+(\alpha, 1, 1)\rangle$ than $|\Psi_+(\alpha, 0, 0)\rangle$. Although the odd entangled coherent state is a maximum entangled one, the fidelity of quantum teleportation by using it as an entangled channel is always smaller than the classical limit value (1/2). Thus it is implied that a maximum entangled state may be not appreciate for the effective quantum teleportation.

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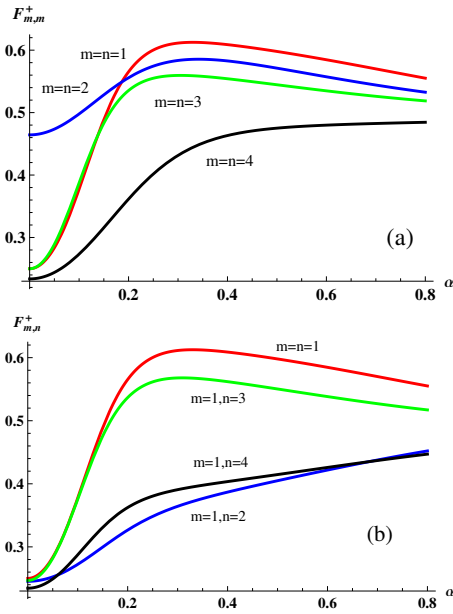


FIG. 7: (Color online) The fidelity as a function of α for several different values of (m, n) , where α is taken a real number and $r = 0.195$.

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